d-dimensional generalization of the point canonical transformation for a quantum particle with position-dependent mass

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Abstract

The d-dimensional generalization of the point canonical transformation for a quantum particle endowed with a position-dependent mass in Schrödinger equation is described. Illustrative examples including; the harmonic oscillator, Coulomb, spiked harmonic, Kratzer, Morse oscillator, Pőschl-Teller and Hulthén potentials are used as *reference* potentials to obtain exact energy eigenvalues and eigenfunctions for *target* potentials at different position-dependent mass settings.

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1 Introduction

A position-dependent effective mass associated with a quantum mechanical particle in the Shrödinger equation have attracted intense research activities over the years [1- 14]. It constitutes an interesting and useful model for the study of many physical problems. In the energy density functional settings to the many-body problem [1], the non-local term of the associated potential can be often expressed as a position-dependence of an appropriate effective mass $M(r)$. Such an effective mass concept is used, for example, in the determination of the electronic properties of the semiconductors [2] and quantum dots [3], in quantum liquids [4], in ^3He clusters [5] and metal clusters [6]. Nevertheless, within the Bohmian approach to quantum theory, the possibility of deriving the Schrödinger equation of particles with position-dependent effective mass from the Riemannian metric structure is explored and discussed (cf., e.g. [7]). Full and partial revivals of a free wave-packet, with position-dependent effective

mass, inside an infinite potential well are studied and documented [8]. N-fold supersymmetry with position-dependent mass was reported by Tanaka [8], etc.

However, in the study of Hamiltonians for particles endowed with position-dependent mass, $M(r) = m_\circ m(r)$, problems of delicate nature erupt in the process. The momentum operator, for example, does not commute with $m(r)$. The choice of the kinetic energy operator is not unique, hence given rise of a quantum mechanical problem of long standing known as ordering ambiguity. Comprehensive details on this issue can be found in the sample of references in [8].

The above have formed, by large, the manifestos/inspirations of the recent studies on the one-dimensional Schrödinger equation for a particle with position-dependent effective mass [7-12]. However, conceptual and fundamental understandings of the quantum physical phenomena may only be enlightened by the exact solvability of the Schrödinger equation. Yet, such exact solutions form the road-map for improving numerical solutions to more complicated physical problems.

On the other hand, it is concreted that exactly solvable problems fall within distinct classes of shape invariant potentials (cf., e.g., [12,13]). Each of which carries a representation of a dynamical group and can be mapped into one another by a point canonical transformation (PCT) (cf., e.g., Alhaidari in [11,12] and Junker in [14]). The Coulomb, the oscillator, and the s-state Morse problems, for example, belong to the shape invariant potentials carrying a representation of $so(2,1)$ Lie algebra. In short, a PCT maintains the canonical form of Schrödinger equation invariant.

In the PCT settings, one needs the exact solution of a potential model in a class of shape invariant potentials to form the so-called *reference potential*. The *reference potential* along with its exact solution (i.e. eigenvalues and eigenfunctions) is then mapped into the so-called *target potential* , hence exact solution for the *target potential* is obtained. Such a recipe is not only bounded to exact solutions but it is also applicable to the quasi-exact and conditionally exact ones (cf, e.g., a sample of references in [15] on the quasi-exact and conditionally exact solutions), a consensus that should remain beyond doubts as long as the canonical form remains invariant.

For the sake of completeness, the PCT approach for a quantum particle with a position-dependent effective mass $M(r) = m_\circ\, m(r)$, in Schrödinger equation, should be complemented by its d-dimensional generalization. Where interdimensional degeneracies associated with the isomorphism between angular momentum ℓ and dimensionality d are incorporated through the central repulsive/attractive core $\ell(\ell+1)/r^2 \longrightarrow \ell_d(\ell_d+1)/r^2$ of the spherically symmetric effective potential $V_{eff}(r) = \ell(\ell+1)/r^2 + V(r)$ (cf, e.g., [16-18] for more details). To the best of our knowledge, the only attempt was made by Gang [18] on an approximate series solutions of the d-dimensional position-dependent mass in Schrödinger equation.

The forthcoming sections are organized as follows. In section 2 we provide the d-dimensional generalization of the point canonical transformation. We discuss the consequences of a power-law radial mass in the same section. Illus-

trative examples are given in section 3. These examples include; the harmonic oscillator, the Coulomb, a spiked harmonic, a Kratzer-molecular, and a Morse oscillator as *reference* potentials. We also give two illustrative examples on the generalized Pöschl-Teller and Hulthén as *reference* potentials with different position-dependent singular masses in section 3. Our concluding remarks are given in section 4.

2 PCT d-dimensional generalization

Following the symmetry ordering recipe of the momentum and position-dependent effective mass ($M(\vec{r}) = m_\circ \, m(\vec{r})$, and $\alpha = \gamma = 0$, and $\beta = -1$ in equation (1.1) of Tanaka in [8]), the Schrödinger Hamiltonian with a potential field $V(\vec{r})$ would read (in atomic units $\hbar = m_\circ = 1$)

$$H = \frac{1}{2}\left(\vec{p}\frac{1}{M(\vec{r})}\right)\cdot\vec{p} + V(\vec{r}) = -\frac{\hbar}{2m_\circ}\left(\vec{\nabla}\frac{1}{m(\vec{r})}\right)\cdot\vec{\nabla} + V(\vec{r}). \tag{1}$$

and assuming the d-dimensional spherical symmetric recipe (cf, e.g., Nieto in [18] for further comprehensive details on this issue), with

$$\Psi(\vec{r}) = r^{-(d-1)/2} R_{n_r,\ell_d}(r)\, Y_{\ell_d,m_d}(\theta,\varphi), \tag{2}$$

Hamiltonian (1) would result in the following time-independent d-dimensional radial Schrödinger equation

$$\left\{\frac{d^2}{dr^2} - \frac{\ell_d(\ell_d+1)}{r^2} + \frac{m'(r)}{m(r)}\left(\frac{d-1}{2r} - \frac{d}{dr}\right) - 2m(r)\left[V(r) - E\right]\right\} R_{n_r,\ell}(r) = 0. \tag{3}$$

Where $\ell_d = \ell + (d-3)/2$ for $d \geq 2$, ℓ is the regular angular momentum quantum number, $n_r = 0, 1, 2, \cdots$ is the radial quantum number, and $m'(r) = dm(r)/dr$. Moreover, the $d = 1$ can be obtained through $\ell_d = -1$ and $\ell_d = 0$ for even and odd parity, $\mathcal{P} = (-1)^{\ell_d+1}$, respectively [17]. Nevertheless, the inter-dimensional degeneracies associated with the isomorphism between angular momentum ℓ and dimensionality d builds up the ladder of excited states for any given n_r and nonzero ℓ from the $\ell = 0$ result, with that n_r, by the transcription $d \to d + 2\ell$. That is, if $E_{n_r,\ell}(d)$ is the eigenvalue in d-dimensions then

$$E_{n_r,\ell}(2) \equiv E_{n_r,\ell-1}(4) \equiv \cdots \equiv E_{n_r,1}(2\ell) \equiv E_{n_r,0}(2\ell+2) \tag{4}$$

for even d, and

$$E_{n_r,\ell}(3) \equiv E_{n_r,\ell-1}(5) \equiv \cdots \equiv E_{n_r,1}(2\ell+1) \equiv E_{n_r,0}(2\ell+3) \tag{5}$$

for odd d. Yet, a unique isomorphism exists between the S-wave ($\ell = 0$) energy spectrum in 3D and in 1D (i.e., $E_{n_r,0}(1) = E_{n_r,0}(3)$). For more details on inter-dimensional degeneracies the reader may refer to, e.g., [16-18].

A substitution of the form $R(r) = m(r)^{\upsilon}\,\phi(Z(r))$ in (3) would result in $Z'(r) = m(r)^{1-2\upsilon}$, manifested by the requirement of a vanishing coefficient of the first-order derivative of $\phi(Z(r))$ (hence a one-dimensional form of Schrödinger equation is achieved), and $Z'(r)^2 = m(r)$ to avoid position-dependent energies. This, in turn, mandates $\upsilon = 1/4$ and suggests the following point canonical transformation

$$q = Z(r) = \int^{r} \sqrt{m(y)}dy \implies \phi_{n_r,\ell_d}(Z(r)) = m(r)^{-1/4} R_{n_r,\ell_d}(r). \tag{6}$$

Which in effect implies

$$\left\{ -\frac{d^2}{dq^2} + \frac{\ell_d(\ell_d+1)}{r^2 m(r)} + 2\left[V(r) - U_d(r) - E_d\right] \right\} \phi_{n_r,\ell_d}(q) = 0, \tag{7}$$

where

$$U_d(r) = \frac{m''(r)}{8m(r)^2} - \frac{7m'(r)^2}{32m(r)^3} + \frac{m'(r)(d-1)}{4r\, m(r)^2}. \tag{8}$$

On the other hand, an exactly solvable (including conditionally-exactly or quasi-exactly solvable) d-dimensional time-independent radial Schrödinger wave equation (with a constant mass $m_\circ$ and cast in $\hbar = m_\circ = 1$ units)

$$\left\{ -\frac{d^2}{dq^2} + \frac{\mathcal{L}_d(\mathcal{L}_d+1)}{q^2} + 2\left[V(q) - \varepsilon\right] \right\} \psi_{n_r,\ell_d}(q) = 0 \tag{9}$$

would form a *reference* for the exact solvability of the *target* equation (7). That is, if the exact/conditionally-exact/quasi-exact solution (analytical/numerical) of (9) is known one can construct the exact/conditionally-exact/quasi-exact solution of (7) through the relation

$$\frac{\ell_d(\ell_d+1)}{2r^2 m(r)} + V(r) - U_d(r) - E \iff \frac{\mathcal{L}_d(\mathcal{L}_d+1)}{2q^2} + V(q) - \varepsilon, \tag{10}$$

Where $\mathcal{L}_d$ is the d-dimensional angular momentum quantum number of the *reference* Schrödinger equation.

2.1 Consequences of a power-law mass $m(r) = \alpha r^{\gamma}$

With the radial position-dependent mass $m(r) = \alpha r^{\gamma}$, the PCT function in (6) implies

$$Z(r) = \sqrt{\alpha}\int^{r} y^{\gamma/2} dy = \frac{2\sqrt{\alpha}}{(\gamma+2)} r^{(\gamma+2)/2} \implies \frac{(\gamma+2)}{2} Z(r) = r\sqrt{m(r)} \tag{11}$$

and (8) gives

$$U_d(r) = -\frac{1}{16}\left(\frac{\gamma(3\gamma+12-8d)}{2r^2 m(r)}\right) \equiv -\frac{1}{4}\left(\frac{\gamma(3\gamma+12-8d)}{2(\gamma+2)^2 Z^2(r)}\right) \tag{12}$$

Relation (10) in effect reads, with $q = Z(r)$,

$$\frac{\tilde{\Lambda}\left(\tilde{\Lambda}+1\right)}{2r^2 m(r)}\left(\frac{\gamma}{2}+1\right)^2 + V(r) - E \iff \frac{\mathcal{L}_d(\mathcal{L}_d+1)}{2q^2} + V(q) - \varepsilon, \tag{13}$$

with

$$\tilde{\Lambda} = -\frac{1}{2} + |\gamma+2|^{-1}\sqrt{4\ell_d(\ell_d+1) + (\gamma-1)^2 + 2\gamma(3-d)} \tag{14}$$

Obviously, Eqs.(11), (13) and (14) suggest that $\gamma = -2$ is not allowed.

2.2 Remedy at $\gamma = -2$ in a power-law mass $m(r) = \alpha r^{\gamma}$

For the case where $m(r) = \alpha r^{-2}$ equation (6) implies

$$q = Z(r) = \sqrt{\alpha}\int^r t^{-1}dt = \sqrt{\alpha}\ln r, \tag{15}$$

and hence

$$\begin{aligned}\tilde{U}_d(\gamma=-2) &= U_d(r,\gamma=-2) - \frac{\ell_d(\ell_d+1)}{2\alpha} \\ &= -\left[\frac{\left(\ell_d+\frac{1}{2}\right)^2 + d - 1}{2\alpha}\right].\end{aligned} \tag{16}$$

Which would only add a constant to the left-hand-side of (10) to yield, with $\mathcal{L}_d = 0$ and/or $\mathcal{L}_d = -1$ (i.e., only s-states and/or $d = 1$ states are available from the right-hand-side of (10)),

$$V(r) - \tilde{U}_d(\gamma=-2) - E \iff V(q) - \varepsilon. \tag{17}$$

3 Illustrative examples

3.1 $m(r) = \alpha r^{\gamma}$ with $\gamma \neq -2$

3.1.1 The harmonic oscillator *reference* potential

The harmonic oscillator,

$$V(q) = \frac{1}{2}\lambda^4 q^2,$$

as a *reference* potential, with the exact d-dimensional eigenenergies and wavefunction

$$\varepsilon_{n_r,\mathcal{L}_d} = \lambda^2\left(2n_r + \mathcal{L}_d + \frac{3}{2}\right), \tag{18}$$

$$\psi_{n_r,\mathcal{L}_d}(q) = a_{n_r,\mathcal{L}_d}\,(\lambda q)^{\mathcal{L}_d+1}\exp\left(-\frac{\lambda^2 q^2}{2}\right)\,L_{n_r}^{\mathcal{L}_d+1/2}\left(\lambda^2 q^2\right) \tag{19}$$

respectively, would imply a *target* potential

$$V\left(r\right)=\frac{\omega^{2}}{2}\alpha r^{\gamma+2};\;\;\omega=\frac{2\lambda^{2}}{\left(\gamma+2\right)},\tag{20}$$

with corresponding d-dimensional eigenenergies and wavefunctions

$$E_{n_r,\ell_d}=\frac{\left(\gamma+2\right)\omega}{2}\left(2n_r+\Lambda+1\right),\tag{21}$$

$$R_{n_r,\ell_d}\left(r\right)=A_{n_r,\ell_d}\;\left(\zeta r\right)^{\left(\frac{\gamma}{2}+1\right)\Lambda+\frac{\left(\gamma+1\right)}{2}}\exp\left(-\frac{\left(\zeta r\right)^{\gamma+2}}{2}\right)L_{n_r}^{\Lambda}\left(\left(\zeta r\right)^{\gamma+2}\right),\tag{22}$$

where

$$\Lambda=\tilde{\Lambda}+1/2=\left|\gamma+2\right|^{-1}\sqrt{4\ell_d\left(\ell_d+1\right)+\left(\gamma-1\right)^{2}+2\gamma\left(3-d\right)}\tag{23}$$

and

$$\zeta=\left[2\alpha\omega/\left(\gamma+2\right)\right]^{1/\left(\gamma+2\right)}\tag{24}$$

It should be noted that this results, at $d=3$, collapse into Alhaidari's ones in example 5(a) of his Appendix in [12], where our ω equals Alhaidari's C.

3.1.2 The Coulomb *reference* potential

The Coulomb,

$$V\left(q\right)=-A/q,$$

as a *reference* potential, with the exact d-dimensional eigenenergies and wavefunction

$$\varepsilon_{n_r,\mathcal{L}_d}=-\frac{\lambda_{n_r,\mathcal{L}_d}^{2}}{8};\;\;\lambda_{n_r,\mathcal{L}_d}=\frac{2A}{\left(n_r+\mathcal{L}_d+1\right)},\tag{25}$$

$$\psi_{n_r,\mathcal{L}_d}\left(q\right)=N_{n_r,\mathcal{L}_d}\;q^{\mathcal{L}_d+1}\exp\left(-\frac{\lambda_{n_r,\mathcal{L}_d}\,q}{2}\right)\;L_{n_r}^{2\mathcal{L}_d+1}\left(\lambda_{n_r,\mathcal{L}_d}\,q\right)\tag{26}$$

respectively, would imply a *target* potential

$$V(r)=-\frac{C}{2\sqrt{\alpha}}\,r^{-1-\gamma/2};\;\;C=A\left(\gamma+2\right),\tag{27}$$

with corresponding d-dimensional eigenenergies and wavefunctions

$$E_{n_r,\ell_d}=-\frac{C^{2}/2}{\left(\gamma+2\right)^{2}}\frac{1}{\left(n_r+\Lambda+1/2\right)^{2}}\tag{28}$$

$$R_{n_r,\ell_d}\left(r\right)=A_{n_r,\ell_d}\;\left(\tilde{\zeta}\,r\right)^{\left(\frac{\gamma}{2}+1\right)\Lambda+\frac{\left(\gamma+1\right)}{2}}\exp\left(-\frac{\left(\tilde{\zeta}\,r\right)^{\frac{\gamma}{2}+1}}{2}\right)L_{n_r}^{2\Lambda}\left(\left(\tilde{\zeta}\,r\right)^{\frac{\gamma}{2}+1}\right)\tag{29}$$

where

$$\tilde{\zeta}=\tilde{\zeta}\left(n_r,\ell_d\right)\;=\left[\frac{4C\sqrt{\alpha}}{\left(\gamma+2\right)^2}\left(n_r+\Lambda+1/2\right)^{-1}\right]^{1/\left(\frac{\gamma}{2}+1\right)} \tag{30}$$

It should be noted that our results (27)-(30), at $d=3$, collapse into Alhaidari's ones reported in example 5(b) of his Appendix in [12], as his second solution of his Eq. (3.3) using the *reference* potential 3D harmonic oscillator. It seems that, in Alhaidari's second solution proposal of his Eq.(3.3) there is an implicit latent additional change of variables of a Liouvillean nature (cf., e.g., [24-26]) that led to a Coulomb-harmonic oscillator correspondence (the reader may wish to investigate this issue following, e.g.,Znojil and Lévai [26]). The proof of which is beyond our current proposal.

3.1.3 A spiked harmonic oscillator *reference* potential

A spiked harmonic oscillator (or a Gold'man and Krivchenkov model),

$$V(q)=\lambda^4q^2/2+\beta q^{-2}/2,$$

as a *reference* potential (cf,e.g.,[20]), with the exact $d-$dimensional eigenenergies and wavefunction

$$\varepsilon_{n_r,\mathcal{L}_d}=\lambda^2\left(2n_r+\tilde{\mathcal{L}}_d+\frac{3}{2}\right),\quad \tilde{\mathcal{L}}_d=-\frac{1}{2}+\sqrt{\left(\mathcal{L}_d+\frac{1}{2}\right)^2+\beta}. \tag{31}$$

$$\psi_{n_r,\mathcal{L}_d}\left(q\right)=a_{n_r,\mathcal{L}_d}\;\left(\lambda q\right)^{\tilde{\mathcal{L}}_d+1}\;\exp\left(-\frac{\lambda^2q^2}{2}\right)\;L_{n_r}^{\tilde{\mathcal{L}}_d+1/2}\left(\lambda^2q^2\right) \tag{32}$$

would lead to a *target* potential

$$V\left(r\right)=\frac{\omega^2}{2}\alpha r^{\gamma+2}+\frac{\tilde{\beta}}{2\alpha}r^{-\gamma-2};\quad \omega=\frac{2\lambda^2}{\left(\gamma+2\right)},\;\tilde{\beta}=\frac{\beta\left(\gamma+2\right)^2}{4} \tag{33}$$

with corresponding d-dimensional eigenenergies and wavefunctions

$$E_{n_r,\ell_d}=\frac{\left(\gamma+2\right)\omega}{2}\left(2n_r+\delta+1\right)\;;\delta=\sqrt{\Lambda^2+\beta} \tag{34}$$

$$R_{n_r,\ell_d}\left(r\right)=N_{n_r,\ell_d}\;\left(\zeta r\right)^{\left(\frac{\gamma}{2}+1\right)\delta+\left(\gamma+1\right)/2}\exp\left(-\frac{\left(\zeta r\right)^{\gamma+2}}{2}\right)L_{n_r}^{\delta}\left(\left(\zeta r\right)^{\gamma+2}\right), \tag{35}$$

It should be noted that Eq.(31) reduces to Eq.(12) of Yu and Dong [10] when $d=1$, $\mathcal{L}_d=0,-1$, and $\gamma=-3$ to read

$$E_{n_r,0}=\lambda^2\left(2n_r+3\right)=\sqrt{2A}\left(2n_r+3\right)$$

where our $\lambda^2 = \sqrt{2A}$, A is defined by Yu and Dong [10] as $A = \xi\tau^2/4$, and our $\beta/2 = 15/8$. One can also show that u in Yu and Dong is equal to $\left[\lambda^2 q^2\right]$, and hence the corresponding wave function, for the $d = 1$ case,

$$\psi_{n_r,\mathcal{L}_d}(q) = N_{n_r}\ \left(\sqrt{u}\right)^{\Lambda+1/2} \exp\left(-\frac{u}{2}\right)\ L_{n_r}^{\Lambda}(u),$$

with $\Lambda = 2$, is exactly the same as that in Eq.(11) of Yu and Dong in [10]. Moreover, our $V(r)$ in (33) is the same as Eq. (4b) Yu and Dongs, of course with the proper amendments.

3.1.4 A Kratzer's-type *reference* potential

A Kratzer's-type molecular potential (cf, e.g., Flügge in [21]),

$$V(q) = -A/q + \beta q^{-2}/2,$$

as a *reference* potential with the exact $d-$dimensional eigenenergies and wave-function

$$\varepsilon_{n_r,\mathcal{L}_d} = -\frac{\tilde{\lambda}^2_{n_r,\mathcal{L}_d}}{8};\ \ \tilde{\lambda}_{n_r,\mathcal{L}_d} = \frac{2A}{\left(n_r + \tilde{\mathcal{L}}_d + 1\right)}, \tag{36}$$

$$\psi_{n_r,\mathcal{L}_d}(q) = N_{n_r,\mathcal{L}_d}\ q^{\tilde{\mathcal{L}}_d+1} \exp\left(-\frac{\tilde{\lambda}_{n_r,\mathcal{L}_d}\, q}{2}\right)\ L_{n_r}^{2\tilde{\mathcal{L}}_d+1}\left(\tilde{\lambda}_{n_r,\mathcal{L}_d}\, q\right) \tag{37}$$

respectively, would imply a set of *target* potentials

$$V(r) = -\frac{C}{2\sqrt{\alpha}}\, r^{-1-\gamma/2} + \frac{\tilde{\beta}}{2\alpha} r^{-\gamma-2}\ , \tag{38}$$

where $C = A(\gamma+2)$, $\tilde{\beta} = \beta(\gamma+2)^2/4$, and corresponding d-dimensional eigenenergies and wavefunctions

$$E_{n_r,\ell_d} = -\frac{C^2/2}{(\gamma+2)^2} \frac{1}{(n_r+\delta+1/2)^2};\quad \delta = \sqrt{\Lambda^2+\beta} \tag{39}$$

$$R_{n_r,\ell_d}(r) = A_{n_r,\ell_d}\ (\eta\, r)^{\left(\frac{\gamma}{2}+1\right)\delta + \frac{(\gamma+1)}{2}} \exp\left(-\frac{(\eta\, r)^{\frac{\gamma}{2}+1}}{2}\right) L_{n_r}^{2\delta}\left((\eta\, r)^{\gamma/2+1}\right) \tag{40}$$

where

$$\eta = \eta(n_r, \ell_d) = \left[\frac{4C\sqrt{\alpha}}{(\gamma+2)^2}(n_r+\delta+1/2)^{-1}\right]^{1/\left(\frac{\gamma}{2}+1\right)} \tag{41}$$

3.2 $m(r) = \alpha r^{\gamma}$ with $\gamma = -2$

3.2.1 The spiked harmonic oscillator *reference* potential

The spiked harmonic oscillator (or a Gold'man and Krivchenkov model),

$$V(q) = \lambda^4 q^2/2 + \beta q^{-2}/2,$$

as a *reference* potential (cf,e.g.,[20]), with the exact $d-$dimensional s-states' eigenenergies and wavefunctions

$$\varepsilon_{n_r,0} = \lambda^2 \left(2n_r + k_d + \frac{3}{2} \right), \;\; k_d = -\frac{1}{2} + \sqrt{\left(\frac{1}{2}\right)^2 + \beta}. \tag{42}$$

$$\psi_{n_r,0}(q) = a_{n_r,0}\, (\lambda q)^{k_d+1}\; e^{-\lambda^2 q^2/2}\; L_{n_r}^{k_d+1/2}\left(\lambda^2 q^2\right) \tag{43}$$

would lead to a *target* potential

$$V(r) = \frac{1}{2\alpha} (\ln r)^2 + \frac{C^2}{2} (\ln r)^{-2}; \quad \alpha = \lambda^{-2},\; C^2 = \frac{\beta}{\alpha} \tag{44}$$

with a corresponding d-dimensional eigenenergies and wavefunctions

$$E_{n_r,\ell_d} = \frac{1}{\alpha} \left(2n_r + \Omega + \frac{\left(\ell_d + \frac{1}{2}\right)^2 + d + 1}{2} \right), \Omega = \; 2^{-1}\sqrt{1 + 4\alpha\, C^2} \tag{45}$$

$$R_{n_r}(r) = B_{n_r}\, \frac{1}{\sqrt{r}}\, (\ln r)^{\Omega+1/2}\; \exp\left[-\frac{(\ln r)^2}{2} \right] L_{n_r}^{\Omega}\left((\ln r)^2 \right), \tag{46}$$

It should be reported here that equations (45) and (46) reduce to the results obtained by Alhaidari (see example 5 in the Appendix of [12]) for $\ell_d = 0$ and $d = 3$, However, it is worthy to mention that the ℓ_d-dependence of the energy eigenvalues of the *target* potential are manifested by the consideration of the constant term in (16) of our proposal.

3.2.2 A Kratzer's-type molecular *reference* potential

A Kratzer's-type molecular potential,

$$V(q) = -A/q + \beta q^{-2}/2,$$

as a *reference* potential with the exact d-dimensional s-states' eigenenergies and wavefunctions

$$\varepsilon_{n_r,0} = -\frac{\tilde{\lambda}_{n_r,0}^2}{8}; \;\; \tilde{\lambda}_{n_r,0} = \frac{2A}{(n_r + k_d + 1)}, \tag{47}$$

$$\psi_{n_r,0}(q) = N_{n_r,0}\; q^{k_d+1}\; \exp\left(-\frac{\tilde{\lambda}_{n_r,0}\, q}{2} \right)\; L_{n_r}^{2k_d+1}\left(\tilde{\lambda}_{n_r,0}\, q \right) \tag{48}$$

would lead to the *target* potential

$$V(r) = -\frac{A}{\sqrt{\alpha}\ln r} + \frac{\beta}{2\alpha(\ln r)^2} \tag{49}$$

with corresponding d-dimensional eigenenergies and wavefunctions

$$E_{n_r,\ell_d} = \frac{\left(\ell_d + \frac{1}{2}\right)^2 + d - 1}{2\alpha} - \frac{\tilde{\lambda}^2_{n_r,0}}{8}, \tag{50}$$

$$R_{n_r}(r) = N_{n_r}\frac{1}{\sqrt{r}}(\ln r)^{k_d+1}\exp\left[-\frac{\tilde{\lambda}_{n_r,0}\sqrt{\alpha}\ln r}{2}\right]$$
$$\times L^{2k_d+1}_{n_r}\left(\lambda_{n_r,0}\sqrt{\alpha}\ln r\right). \tag{51}$$

3.2.3 A Morse-oscillator *reference* potential

A Morse-oscillator potential of the form

$$V(q) = Ae^{-2aq} - Be^{-aq}\ ;\quad B = 2A,$$

as a *reference* potential (cf, e.g., [19]) with $\mathcal{L}_d = 0$ and/or $\mathcal{L}_d = -1$,with the exact s-states d-dimensional eigenenergies wavefunctions

$$\varepsilon_{n_r} = -A\left[1 - \sqrt{\frac{1}{2A\alpha}}\left(n_r + \frac{1}{2}\right)\right]^2\ ;\quad a = 1/\sqrt{\alpha} \tag{52}$$

$$\psi_{n_r}(q) = N_{n_r}\,u^s\,e^{-u/2}\,F(-n_r, 2s+1, u)\,; \tag{53}$$
$$u = \sqrt{8\alpha A}e^{-aq},\ s = \sqrt{-2\alpha\varepsilon_{n_r}}.$$

would lead to a *target* potential

$$V(r) = -A\left(\frac{1}{r^2} - \frac{2}{r}\right) \tag{54}$$

with corresponding d-dimensional eigenenergies and wavefunctions

$$E_{n_r,\ell_d} = \frac{\left(\ell_d + \frac{1}{2}\right)^2 + d - 1}{2\alpha} - A\left[1 - \frac{1}{\sqrt{2A\alpha}}\left(n_r + \frac{1}{2}\right)\right]^2 \tag{55}$$

$$R_{n_r}(r) = \tilde{N}_{n_r}\left(\frac{1}{r}\right)^{s+1/2} e^{-u/2}F(-n_r, 2s+1, u)\,. \tag{56}$$

Where $u = \sqrt{8\alpha A}/r$. It should reported here that when $d = 1$ and $\ell_d = 0, -1$ Eq.(55) reduces to Eq.s (14) and (!5) in [10].

3.3 Two example on $m(r) \neq \alpha r^{\gamma}$

3.3.1 A Periodic Generalized Pöschl-Teller *reference* potential and $m(r) = \alpha/4r(1+r)^2 \implies q(r) = \sqrt{\alpha}\arctan\sqrt{r}$

A Generalized Pöschl-Teller potential [21] of the form

$$V(q) = \frac{\zeta^2}{2}\left[\frac{\tau(\tau-1)}{\cos^2\zeta q} + \frac{\varkappa(\varkappa-1)}{\sin^2\zeta q}\right] \tag{57}$$

as a *reference* potential with the exact d-dimensional s-states' eigenvalues and eigenfunctions

$$\varepsilon_{n_r,0} = \frac{1}{2}\zeta^2\left[\varkappa+\tau+2n_r\right]^2 \tag{58}$$

$$\psi_{n_r}(q) = C_{n_r,\varkappa,\tau}\left(\sin\zeta q\right)^{\varkappa}\left(\cos\zeta q\right)^{\tau}\,{}_2F_1\left(-n_r,\varkappa+\tau+n_r,\varkappa+1/2;\sin^2\zeta q\right) \tag{59}$$

would lead to a *target* potential

$$V(r) = V_1\left(1+r^2\right) + V_2\left(1+\frac{1}{r^2}\right) \tag{60}$$

with d-dimensional s-states' eigenenergies and eigenfunctions

$$E_{n_r,\ell_d} = \frac{1}{2}\zeta^2\left[\varkappa+\tau+2n_r\right]^2 - \left(\eta_1+\eta_2+\eta_3\right) \tag{61}$$

$$\begin{aligned} R_{n_r}(r) = C_{n_r,\varkappa,\tau}&\left[\frac{\alpha}{4r(1+r)^2}\right]^{\frac{1}{4}}\left[\frac{r}{1+r}\right]^{\frac{\varkappa}{2}} \\ &\times\left[\frac{1}{1+r}\right]^{\tau}\,{}_2F_1\left(-n_r,\varkappa+\tau+n_r,\varkappa+1/2;\frac{r}{1+r}\right). \end{aligned} \tag{62}$$

Where

$$\begin{gathered} \zeta = 1/\sqrt{\alpha}, \eta_1 = (24d-9)/8\alpha, \eta_2 = (8d-9)/8\alpha, \\ \eta_3 = -(32d-22)/8\alpha \end{gathered} \tag{63}$$

and

$$\begin{aligned} V_1 &= \frac{\varkappa(\varkappa-1) - \left(2\alpha\eta_1 + 4\ell_d(\ell_d+1)\right)}{2\alpha} \\ V_2 &= \frac{\tau(\tau-1) - \left(2\alpha\eta_2 + 4\ell_d(\ell_d+1)\right)}{2\alpha} \end{aligned} \tag{64}$$

3.3.2 A Generalized Hulthén *reference* potential and $m(r) = 1/\alpha^2(r+1)^2 \implies q(r) = \alpha^{-1}\ln(r+1)$

A Generalized Hulthén potential [22] of the form

$$V(q) = -\frac{\alpha e^{-\alpha q}}{1-e^{-\alpha q}} \tag{65}$$

as a *reference* potential with the exact d-dimensional eigenvalues and eigenfunctions (for the s -states)

$$\varepsilon_{n_r,0} = \frac{\alpha^2}{2}Q_{n_r}^2 \tag{66}$$

$$\psi_{n_r}(q) = C_{n_r,0}e^{-Q_{n_r}\alpha q}\sum_{\nu=1}^{n_r}(-1)^{\nu-1}\binom{n_r-1}{\nu-1}\binom{n_r+\beta_{n_r}+\nu-2}{\nu}\left(1-e^{-\alpha q}\right)^{\nu} \tag{67}$$

where $Q_{n_r} = \frac{1}{2}(\frac{2}{n_r\alpha}-n_r)$, and $\beta_{n_r} = 1+2Q_{n_r}$ would leads to a target potential

$$V(r) = -\frac{\sigma}{r} \quad \text{where } \sigma = \alpha(\alpha(d-1)/2+1) \tag{68}$$

with d-dimensional s -states' eigenenergies and eigenfunctions

$$E_{n_r,0} = \frac{\alpha^2}{2}Q_{n_r}^2 + \mathcal{B} \quad \text{where } \mathcal{B} = \alpha^2(4d-3)/8 \tag{69}$$

$$R_{n_r}(r) = C_{n_r}(1+r)^{-Q_{n_r}-1/2}\sum_{\nu=1}^{n_r}(-1)^{\nu-1}\binom{n_r-1}{\nu-1}\binom{n_r+\beta_{n_r}+\nu-2}{\nu}(1-r)^{\nu} \tag{70}$$

4 Concluding Remarks

In the point canonical transformation (PCT) method [23] (an old Liouvillean change of variables spirit [24,25]) a Schrödinger-type equation often mediates (via the existence of invertible parametrization of the real coordinates, $r \longrightarrow r(q)$, and its few derivatives $r'(q)$, $r''(q), \cdots$) a transition between two different effective potentials. In such settings, explicit correspondence (cf, e.g., Znojil and Lévai [26]) between two bound state problems (i.e., the *reference/old* and the *target/new*) is obtained. Within these Liouvillean change of variables' spiritual lines, Alhaidari [11,12] has developed PCT-maps into *target/new* position-dependent effective mass problems, in $d=1$ and $d=3$.

In this paper, a d-dimensional generalization of the PCT method for a quantum particle endowed with a position-dependent mass in Schrödinger equation is described. Our illustrative examples include; the harmonic oscillator, Coulomb,

spiked harmonic oscillator, Kratzer-type molecular, Morse oscillator, Pöschl-Teller and Hulthén potentials as *reference/old* potentials to obtain exact energy eigenvalues and eigenfunctions for *target/new* potentials with different position-dependent effective mass settings.

Finally, the applicability of the current PCT d-dimensional generalization extends beyond the attendant Hermiticity settings to, feasibly, cover not only $\mathcal{PT}$-symmetric non-Hermitian Hamiltonians but also a broader class of η-pseudo Hermitian non-Hermitian Hamiltonians [25-28]. This is already done in [29].

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